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ABSTRACT

We consider a class of Dynkin games in the case where the underlying process evolves according to a one-dimensional but otherwise general diffusion. We establish general conditions under which both the value and the saddle point equilibrium exist and under which the exercise boundaries characterizing the saddle point strategy can be explicitly characterized in terms of a pair of standard first order necessary conditions for optimality. We also analyze those cases where an extremal pair of boundaries exists and show that there are circumstances under which increased volatility may break up the existence of a saddle point.

JEL Classification: C73, C72, C61.

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1 Introduction

Kifer (2000) introduced a new class of derivative securities (Israeli options) which are not only based on the evolution of the underlying asset price but depend on the strategic interaction between the issuer and the holder of the contract as well. The main difference of these derivative securities with the standard American contingent claims is that these contracts can be exercised early by both the issuer as well as the holder. If the issuer decides to exercise the contract early, then the holder receives a higher payoff (i.e. the issuer has to pay a penalty). If the opposite happens, then the holder receives a lower exercise payoff which can be interpreted as a cost reduction for the issuer. Kifer (2000) demonstrated that the valuation of these contracts can be interpreted in terms of the saddle point equilibrium of an associated Dynkin game (cf. Dynkin (1969)). More recently, Kyprianou (2004) analyzed two examples of perpetual Israeli options (δ-penalty put option and the δ-penalty Russian option) and derived their values explicitly. These studies naturally raise several interesting questions on the extent to which the valuation of these contracts can be generalized. Moreover, the comparative static properties of the optimal strategies and their impact on the value of the contracts is naturally also of interest. Our purpose is to address these questions in this study within a relatively general setting.

There are several mathematical techniques for analyzing optimal stopping games based on continuous diffusion models for the underlying stochastic dynamics. Friedman (1973a) considered a broad class of zero-sum stopping games and analyzed their values by relying on variational inequalities (see also Friedman (1973b)). Bensoussan and Friedman (1974, 1977) investigated stopping games and their values in a very general setting both in the nonzero-sum as well as in the zero-sum case by relying on the connection between the value of the game and quasi-variational inequalities. Karatzas and Wang (2001), Fukushima and Taksar (2002), as well as Boetius (2005) analyzed Dynkin games by relying on the connection between singular stochastic control and optimal stopping. In that setting the value characterizing the saddle point equilibrium can be identified as the derivative of the value function of the singular control problem with respect to the current state. Ekström (2006) as well as Ekström and Villeneuve (2006), in turn, analyzed Dynkin games by relying on the relationship between functional concavity and r-excessivity along the lines of the pioneering work by Dynkin (1965) (Chapters XV and XVI) and the subsequent research by Dayanik and Karatzas (2003) on the optimal stopping applications of functional concavity.

In this paper our objective is to analyze a relatively broad class of perpetual Dynkin stopping games where the underlying stochastic dynamics is modeled as a one-dimensional but otherwise general diffusion process. Instead of tackling the problem via variational inequalities or functional concavity, we take an alternative route and analyze the value of the stopping game by relying on the classical theory of diffusions and deriving first the value of an arbitrary stopping policy which can be characterized as a first exit time from an open but otherwise general subinterval of the state space of the underlying diffusion. Naturally, the resulting functional obtained by relying on this approach is not only a function of the minimal r-excessive functions for the underlying diffusion, it is a function of the two arbitrary boundaries as well. In light of this observation, we investigate if these potentially suboptimal boundaries characterizing a pair of stopping times which constitute a candidate pair for the saddle point strategy can be chosen so as to make the representation extremal. By relying on ordinary nonlinear programming techniques, we state a set of ordinary first order necessary conditions characterizing an interior pair of boundaries yielding an extremal representation. We establish a general set of conditions under which these first order conditions admit a unique solution and prove that whenever a pair satisfying the optimality conditions exist, it characterizes the value of the game and, therefore, the saddle point equilibrium strategy as well. Our results demonstrate that strategic interaction accelerates rational exercise in comparison with the non-strategic case by resulting into a continuation region which is typically strictly included into the interception of the continuation regions of the stopping problems associated with the non-strategic setting. Consequently, our results unambiguously indicate that along the saddle point equilibrium both players follow a rule which would be considered suboptimal in the absence of strategic interaction. We also consider the impact of optional components on the optimal policy and find that under certain conditions there are circumstances under which increased volatility (or other similar parametric changes resulting in an expanded continuation set) may break up the existence of a saddle point equilibrium and result only into a suboptimal corner solution. Thus, within our modeling framework increased market instability does not only alter the equilibrium strategy, it may very well abolish it altogether. This observation once again emphasizes the detrimental impact of increased volatility on the optimal timing of exercise in the strategic setting as well.

The contents of this study are as follows. In section two we characterize the underlying diffusion and present the considered class of Dynkin games. In section three we present a set of auxiliary results needed later in the analysis of the general problem. In section four we investigate the saddle point strategy and state our main results on the existence of an optimal pair of stopping boundaries and the resulting value of the game. In section five we then illustrate our general results in two cases. Finally, section six concludes our study.

2 The Considered Dynkin Game

We assume that the underlying state variable evolves according to a linear, time homogeneous, and regular diffusion process defined on the complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ and evolving on the state-space $\mathcal{I} = (a, b) \subseteq \mathbb{R}$ according to the dynamics described by the Itô-stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{1}$$

where both the drift coefficient $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and the diffusion coefficient $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are assumed to be sufficiently smooth for guaranteeing the existence and uniqueness of a (weak) solution for the stochastic differential equation (1) (at least continuous, cf. Borodin and Salminen (1996), pp. 46–47), and W_t denotes standard Brownian motion. In order to avoid interior singularities, we also assume that the diffusion coefficient $\sigma(x)$ is positive, that is, we assume that $\sigma(x) > 0$ for all $x \in \mathcal{I}$. Given the regularity of the underlying diffusion process, we assume that the boundaries of the state-space are either natural, entrance, exit, or regular for the process X_t. In case they are regular, we assume that they are killing. As usually,

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

denotes the differential operator associated to the underlying diffusion X_t. Since we are going to introduce discounting into the analysis of the problem, we define the differential operator $\mathcal{G}_r$ as $\mathcal{G}_r = \mathcal{A} - r$, where $r > 0$ denotes the constant discount rate.

Given the diffusion X_t, denote now as $\mathcal{L}^1$ the class of measurable mappings satisfying the absolute integrability condition (i.e the *absence of speculative bubbles condition*)

$$\mathbf{E}_x \int_0^\zeta e^{-rt} |f(X_t)| dt < \infty,$$

where $\zeta = \inf\{t \geq 0 : X_t \notin \mathcal{I}\}$ denotes the potentially infinite life-time of the underlying diffusion. For any $f \in \mathcal{L}^1$ we define the functional $R_r f : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$(R_r f)(x) = \mathbf{E}_x \int_0^\zeta e^{-rt} f(X_t) dt.$$

Moreover, it is well-known that given the assumptions of our study, there are two linearly independent *fundamental solutions* $\psi(x)$ and $\varphi(x)$ satisfying a set of appropriate boundary conditions based on the boundary behavior of the process X and spanning the set of solutions of the ordinary differential equation $(\mathcal{G}_r u)(x) = 0$ (cf. Borodin and Salminen 2002, pp. 18 - 19). Moreover, $\psi'(x)\varphi(x) - \varphi'(x)\psi(x) = BS'(x)$, where $B > 0$ denotes the constant Wronskian of the fundamental solutions $\psi(x)$ and $\varphi(x)$ and

$$S'(x) = \exp\left(-\int \frac{2\mu(x)dx}{\sigma^2(x)}\right)$$

denotes the density of the scale function of X. Given these fundamental solutions, the expected cumulative present value $(R_r f)(x)$ of a cash flow $f \in \mathcal{L}^1$ can be re-expressed as (for a comprehensive characterization of the fundamental solutions and the representation of expected cumulative present values in terms of these fundamental solutions, see Borodin and Salminen (2002), pp. 17–20 and p. 29)

$$(R_r f)(x) = B^{-1}\varphi(x) \int_a^x \psi(y) f(y) m'(y) dy + B^{-1}\psi(x) \int_x^b \varphi(y) f(y) m'(y) dy, \tag{2}$$

where $m'(y) = 2/(\sigma^2(x)S'(x))$ denotes the density of the speed measure of the underlying state process X_t.

Having characterized the underling diffusion process it is our purpose in this paper to study a infinite horizon Dynkin game characterized by the mapping (we follow the notation of the paper by Ekström and Villeneuve (2006))

$$\Pi_x(\tau,\gamma) = \mathbf{E}_x\left[e^{-r(\tau\wedge\gamma)}\left(g_1(X_\tau)1_{\{\tau\leq\gamma\}} + g_2(X_\gamma)1_{\{\tau>\gamma\}}\right)\right], \tag{3}$$

where $g_i : \mathcal{I} \mapsto \mathbb{R}$, $i = 1, 2$, are continuous, non-decreasing mappings satisfying the condition $g_2(x) \geq g_1(x)$ for all $x \in \mathcal{I}$. We also assume that $g_1(x)$ is bounded from below and that both mappings are continuously differentiable on $\mathcal{I}$ and twice continuously differentiable outside a countable set of points $\{y_i\}_{i\in I}$ so that $|g_j''(y_i\pm)| < \infty$ for $i = 1, 2$ and $j \in I$. The associated lower and upper values are defined as

$$\underline{V}(x) = \sup_\tau \inf_\gamma \Pi_x(\tau,\gamma) \tag{4}$$

and

$$\overline{V}(x) = \inf_\gamma \sup_\tau \Pi_x(\tau,\gamma). \tag{5}$$

As was argued in Ekström and Villeneuve (2006) we naturally have

$$g_1(x) \leq \underline{V}(x) \leq \overline{V}(x) \leq g_2(x).$$

However, if we also have $\underline{V}(x) \geq \overline{V}(x)$, then the stochastic game has a value and this value is denoted as $V(x) = \underline{V}(x) = \overline{V}(x)$. A pair of stopping times (τ', γ') constitutes a saddle point of the considered Dynkin game whenever the condition $\Pi_x(\tau,\gamma') \leq \Pi_x(\tau',\gamma') \leq \Pi_x(\tau',\gamma)$ is satisfied for all stopping times τ, γ. In light of this inequality, it is clear that the existence of a saddle point guarantees the existence of the value for the considered game. Finally, if the considered Dynkin game has the value $V(x)$, then the pair of stopping times

$$\tau^* = \inf\{t \geq 0 : V(X_t) \leq g_1(X_t)\} \tag{6}$$

and

$$\gamma^* = \inf\{t \geq 0 : V(X_t) \geq g_2(X_t)\} \tag{7}$$

constitute a saddle point for the game.

3 Auxiliary Results

Before presenting our first auxiliary results, we define the linear operators L_ψ and L_φ as (cf. Salminen (1985) and Alvarez (2004))

$$(L_\varphi f)(x) = \frac{f'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}f(x) \tag{8}$$

and

$$(L_\psi f)(x) = \frac{f'(x)}{S'(x)}\psi(x) - \frac{\psi'(x)}{S'(x)}f(x). \tag{9}$$

Naturally, if the mapping $f(x)$ is twice continuously differentiable on $\mathcal{I}$, then $(L_\psi f)'(x) = (\mathcal{G}_r f)(x)\psi(x)m'(x)$ and $(L_\varphi f)'(x) = (\mathcal{G}_r f)(x)\varphi(x)m'(x)$. Hence, we observe that the functionals $(L_\psi f)(x)$ and $(L_\varphi f)(x)$ remain constants on the sets where the mapping $f(x)$ is r-harmonic. Similarly, we also observe that the functionals $(L_\psi f)(x)$ and $(L_\varphi f)(x)$ decrease (increase) on the sets where the mapping $f(x)$ is r-superharmonic (r-subharmonic). Given these definitions, we now present a set of results characterizing the optimal stopping strategy in the absence of strategic interaction (cf. Alvarez (2001, 2004)).

Lemma 3.1. *(A) Assume that a is a natural boundary for X_t, that b is a nonattracting boundary for X_t, and that there is a threshold $\tilde{x}_i$, such that $(\mathcal{G}_r g_i)(x) \gtreqless 0$ for $x \lesseqgtr \tilde{x}_i$, $i = 1, 2$. Then*

$$U_i(x) = \inf_\gamma \mathbf{E}_x\left[e^{-r\gamma}g_i(X_\gamma)\right] = \varphi(x)\inf_{y\leq x}\left[\frac{g_i(y)}{\varphi(y)}\right] = \begin{cases} g_i(\hat{x}_i)\frac{\varphi(x)}{\varphi(\hat{x}_i)} & x \in (\hat{x}_i, b) \\ g_i(x) & x \in (a, \hat{x}_i] \end{cases} \tag{10}$$

where the exercise threshold $\hat{x}_i = \operatorname{argmin}\{g_i(x)/\varphi(x)\} \in (a, \tilde{x}_i)$ is the unique root of the ordinary first order condition $g_i'(\hat{x}_i)\varphi(\hat{x}_i) = g_i(\hat{x}_i)\varphi'(\hat{x}_i)$. Moreover, $\gamma_{\hat{x}_i} = \inf\{t \geq 0 : X_t \leq \hat{x}_i\}$ is an almost surely finite optimal stopping time.

(B) Assume that b is a natural boundary for X_t, that a is a nonattracting boundary for X_t, and that there is a threshold $\tilde{x}_i$, such that $(\mathcal{G}_r g_i)(x) \gtreqless 0$ for $x \lesseqgtr \tilde{x}_i$. Then

$$J_i(x) = \sup_\tau \mathbf{E}_x\left[e^{-r\tau}g_i(X_\tau)\right] = \psi(x)\sup_{y\geq x}\left[\frac{g_i(y)}{\psi(y)}\right] = \begin{cases} g_i(x) & x \in [x_i^*, b) \\ g_i(x_i^*)\frac{\psi(x)}{\psi(x_i^*)} & x \in (a, x_i^*) \end{cases} \tag{11}$$

where the exercise threshold $x_i^* = \operatorname{argmax}\{g_i(x)/\psi(x)\} \in (\tilde{x}_i, b)$ *is the unique root of the ordinary first order condition* $g_i'(x_i^*)\psi(x_i^*) = g_i(x_i^*)\psi'(x_i^*)$. *Moreover,* $\tau_{x_i^*} = \inf\{t \geq 0 : X_t \geq x_i^*\}$ *is an almost surely finite optimal stopping time.*

Proof. (A) Consider first the functional $(L_\varphi g_i)(x)$, where the operator L_φ is defined as in (8). The monotonicity of the decreasing fundamental solution $\varphi(x)$ and the exercise payoff $g_i(x)$ implies that $(L_\varphi g_i)(x) > 0$ on $g_i^{-1}(\mathbb{R}_+)$. Moreover, standard differentiation yields

$$(L_\varphi g_i)'(x) = (\mathcal{G}_r g_i)(x)\varphi(x)m'(x) \gtreqqless 0, \quad x \lesseqqgtr \tilde{x}_i.$$

Let $z \in g_i^{-1}(\mathbb{R}_+) \cap (\tilde{x}_i, b)$. Standard integration yields

$$(L_\varphi g_i)(x) = (L_\varphi g_i)(z) - \int_x^z (\mathcal{G}_r g_i)(y)\varphi(y)m'(y)dy.$$

Choosing $x < k < \tilde{x}_i$ and applying the mean value theorem for integrals yields that

$$(L_\varphi g_i)(x) = (L_\varphi g_i)(k) - \int_x^k (\mathcal{G}_r g_i)(y)\varphi(y)m'(y)dy = (L_\varphi g_i)(k) - \frac{(\mathcal{G}_r g_i)(\xi)}{r}\left(\frac{\varphi'(k)}{S'(k)} - \frac{\varphi'(x)}{S'(x)}\right),$$

where $\xi \in (x, k)$. Since $\varphi'(x)/S'(x) \downarrow -\infty$ as $x \downarrow a$ when a is either exit or natural, we find that $(L_\varphi g_i)(x) \downarrow -\infty$ as $x \downarrow a$. Combining this observation with the continuity and monotonicity of $(L_\varphi g_i)(x)$ then proves that there is a unique threshold $\hat{x}_i < \tilde{x}_i$ such that $(L_\varphi g_i)(x) \lesseqqgtr 0$ when $x \lesseqqgtr \hat{x}_i$. Especially, the definition of $(L_\varphi g_i)(x)$ implies that $\hat{x}_i = \operatorname{argmin}\{g_i(x)/\varphi(x)\}$.

Given the above observation, denote now the proposed value function as $\hat{U}_i(x)$. It is now clear from the analysis above that the proposed value function is almost everywhere twice continuously differentiable, that $\hat{U}_i(x) \leq g_i(x)$ for all $x \in \mathcal{I}$, and that $(\mathcal{G}_r \hat{U}_i)(x) = 0$ on $(\hat{x}_i, b)$. However, since $\hat{x}_i < \tilde{x}_i$ we find that $(\mathcal{G}_r \hat{U}_i)(x) \geq 0$ on $(a, \hat{x}_i)$ and, therefore, that $\hat{U}_i(x)$ constitutes a r-subharmonic minorant of $g_i(x)$. Since $U_i(x)$ is the greatest of such minorants, we find $\hat{U}_i(x) \leq U_i(x)$.

To prove the opposite inequality, we first observe that since b is nonattracting, we know that $\mathbb{P}_x[\gamma_y < \infty] = 1$ for any $x \in (y, b)$ where $y > a$ (cf. Karlin and Taylor (1981), p. 226–229). Combining this observation with the continuity of the underlying diffusion process and the exercise payoff implies that the proposed value function can be expressed as

$$\hat{U}_i(x) = \mathbf{E}_x\left[e^{-r\gamma_{\hat{x}_i}} g_i(X_{\gamma_{\hat{x}_i}})\right],$$

where $\gamma_{\hat{x}_i} = \inf\{t \geq 0 : X_t \leq \hat{x}_i\}$. Hence, $\hat{U}_i(x)$ is attained by applying an admissible stopping strategy and, therefore, we find that $\hat{U}_i(x) \geq U_i(x)$, which completes the proof of the alleged result. Proving part (B) is entirely analogous. $\square$

The proof of Lemma 3.1 states a set of conditions under which the two associated stopping problems are solvable in the non-strategic case. As is clear from the proof of Lemma 3.1 the boundary behavior of the underlying diffusion plays an important role in the proof of existence and uniqueness of an optimal stopping boundary. The conditions on the admissible boundary behavior can be, however, relaxed by making extra assumptions on the extreme points of the auxiliary functions $g_i(x)/\psi(x)$ and $g_i(x)/\varphi(x)$ (which are closely related to Doob's ψ-transform (φ-transform, respectively) of the underlying diffusion X; cf. Borodin and Salminen (2002), pp. 33–34). Our main findings on this case is summarized in the next Corollary extending the results of Lemma 3.1 to a broader class of admissible diffusion processes.

Corollary 3.2. *(A) Assume that b is a nonattracting boundary for X_t, that there is a threshold $\hat{x}_i$ such that $(L_\varphi g_i)(x) \gtreqqless 0$ for $x \gtreqqless \hat{x}_i$, $i = 1, 2$, and that $(L_\varphi g_i)'(x) \geq 0$ on $(a, \hat{x}_i)$. Then the conclusions of part (A) of Lemma 3.1 are valid.*
(B) Assume that a is a nonattracting boundary for X_t, that there is a threshold x_i^* such that $(L_\psi g_i)(x) \gtreqqless 0$ for $x \lesseqqgtr x_i^*$, and that $(L_\psi g_i)'(x) \leq 0$ on (x_i^*, b). Then the conclusions of part (B) of Lemma 3.1 are valid.

Proof. The alleged results are direct implications of Lemma 3.1. $\square$

Corollary 3.2 essentially demonstrates that if the existence and uniqueness of the extreme points of the functions $g_i(x)/\psi(x)$ and $g_i(x)/\varphi(x)$ are given and a set of sufficient monotonicity conditions for the functionals $(L_\varphi g_i)(x)$ and $(L_\psi g_i)(x)$ are satisfied, then the conclusions of Lemma 3.1 can be extended for a more general boundary behavior of the underlying diffusion. As we will later see in our analysis, this finding extends to the strategic setting as well.

It is clear from the definition of the saddle point characterized by the stopping times (6) and (7) that the value of the game constitutes the smallest r-superharmonic majorant of the payoff $g_1(x)$ on the set where $V(x) \leq g_2(x)$ and the greatest r-subharmonic minorant of the payoff $g_2(x)$ on the set where $V(x) \geq g_1(x)$. On the other hand, the r-harmonicity of the value function on the continuation region $C = \{x \in \mathcal{I} : g_1(x) < V(x) < g_2(x)\}$ where exercising is suboptimal implies that if $x \in (z, y) \subset C$ and $a < z < y < b$ then

$$V(x) = \mathbf{E}_x[e^{-r(\lambda_z \wedge \lambda_y)} V(X_{\lambda_z \wedge \lambda_y})] \tag{12}$$

where $\lambda_l = \inf\{t \geq 0 : X_t = l\}$ denotes the first hitting time of the underlying diffusion to the state $l \in \mathcal{I}$. As is well-known from the classical theory of diffusions (12) can be re-expressed in terms of the minimal r-excessive mappings $\psi(x)$ and $\varphi(x)$ as

$$V(x) = V(z)\frac{\hat{\varphi}(x)}{\hat{\varphi}(z)} + V(y)\frac{\hat{\psi}(x)}{\hat{\psi}(y)}, \tag{13}$$

where

$$\hat{\psi}_z(x) = \psi(x) - \frac{\psi(z)}{\varphi(z)}\varphi(x)$$

denotes the increasing and

$$\hat{\varphi}_y(x) = \varphi(x) - \frac{\varphi(y)}{\psi(y)}\psi(x)$$

denotes the decreasing fundamental solutions (unique up to a multiplicative constant) of the ordinary differential equation $(\mathcal{A}u)(x) = ru(x)$ defined on the domain of the operator of the killed diffusion $\{X_t; t \in [0, \lambda_z \wedge \lambda_y)\}$. Naturally, $\hat{\psi}_z(x) \uparrow \psi(x)$ as $z \downarrow a$ and $\hat{\varphi}_y(x) \uparrow \varphi(x)$ as $y \uparrow b$. It is now clear in light of this representation that an interesting question is whether there is a pair of constant boundaries at which this expression is extremal. Moreover, given that such a pair exists it is naturally of interest to study whether the resulting stopping strategy constitutes an equilibrium strategy of the game. We address this problem in the next section.

4 The Saddle Point Solution

It is clear from the r-harmonicity of the value function on the continuation region and representation (13) that in the general case two different cases may arise. Namely, the case where the free boundary problem results into a stopping rule which can be characterized by a single threshold and the case where the saddle point strategy can be characterized by two boundaries at which the underlying diffusion should be stopped. Since the buyers objective is to maximize the exercise payoff, the determination of the upper boundary y characterizes the best possible solution for the buyer in the case where the buyer has to settle for the lower payoff. Similarly, since the objective of the seller is to minimize the exercise payoff received by the buyer, the determination of the lower boundary z characterizes the best possible solution for the seller in the case where the buyer obtains the higher payoff. Given the characterization (13) consider now the mapping

$$F_{z,y}(x) = g_2(z)\frac{\hat{\varphi}_y(x)}{\hat{\varphi}_y(z)} + g_1(y)\frac{\hat{\psi}_z(x)}{\hat{\psi}_z(y)} \tag{14}$$

which coincides with the value $V(x)$ whenever the identities $V(z) = g_2(z)$ and $V(y) = g_1(y)$ are satisfied in (13). Given this auxiliary expression, it is now economically sensible to ask whether this representation attains an extreme value as a function of the arbitrary pair of boundaries z and y for any $x \in (z, y)$. Our first set of results summarizing the principal implications on this question are now presented in the following.

Lemma 4.1. *The boundaries z^*, y^* satisfy the conditions*

$$\begin{aligned} \frac{g_2'(z^*)}{S'(z^*)}\hat{\varphi}_{y^*}(z^*) - \frac{\hat{\varphi}_{y^*}'(z^*)}{S'(z^*)}g_2(z^*) &= B\frac{g_1(y^*)}{\psi(y^*)} && (15)\\ \frac{g_1'(y^*)}{S'(y^*)}\hat{\psi}_{z^*}(y^*) - \frac{\hat{\psi}_{z^*}'(y^*)}{S'(y^*)}g_1(y^*) &= -B\frac{g_2(z^*)}{\varphi(z^*)} && (16) \end{aligned}$$

which can be re-expressed as

$$\begin{aligned} B^{-1}\int_{z^*}^{y^*}(\mathcal{G}_r g_2)(x)\psi(y^*)\hat{\varphi}_{y^*}(x)m'(x)dx &= g_2(y^*) - g_1(y^*) && (17)\\ B^{-1}\int_{z^*}^{y^*}(\mathcal{G}_r g_1)(x)\varphi(z^*)\hat{\psi}_{z^*}(x)m'(x)dx &= g_1(z^*) - g_2(z^*). && (18) \end{aligned}$$

Moreover, if a pair z^*, y^* satisfying the first order necessary conditions (15) *and* (16) *exists, then the value $F_{z^*,y^*}(x)$ satisfies the smooth-fit conditions $F_{z^*,y^*}'(z^*) = g_2'(z^*)$ and $F_{z^*,y^*}'(y^*) = g_1'(y^*)$.*

Proof. Since

$$\frac{\partial}{\partial z}\hat{\psi}_z(x) = -\frac{BS'(z)}{\varphi^2(z)}\varphi(x) \quad \text{and} \quad \frac{\partial}{\partial z}\hat{\varphi}_y(x) = \frac{BS'(y)}{\psi^2(y)}\psi(x)$$

we find by standard differentiation of the value $F_{z,y}(x)$ with respect to the boundaries z and y that if an extremal pair exists it has to satisfy the ordinary first order necessary conditions

$$\begin{aligned}\frac{\partial F_{z,y}}{\partial z}(x) &= \frac{\hat{\varphi}_y(x)S'(z)}{\hat{\varphi}_y^2(z)}\left[\frac{g_2'(z)}{S'(z)}\hat{\varphi}_y(z) - \frac{\hat{\varphi}_y'(z)}{S'(z)}g_2(z) - B\frac{g_1(y)}{\psi(y)}\right] = 0\\ \frac{\partial F_{z,y}}{\partial y}(x) &= \frac{\hat{\psi}_z(x)S'(y)}{\hat{\psi}_z^2(y)}\left[\frac{g_1'(y)}{S'(y)}\hat{\psi}_z(y) - \frac{\hat{\psi}_z'(y)}{S'(y)}g_1(y) + B\frac{g_2(z)}{\varphi(z)}\right] = 0\end{aligned}$$

from which (15) and (16) follow. The alternative representations (17) and (18) follow from (15) and (16) by noticing that

$$\frac{d}{dx}\left[\frac{g_2'(x)}{S'(x)}\hat{\varphi}_y(x) - \frac{\hat{\varphi}_y'(x)}{S'(x)}g_2(x)\right] = (\mathcal{G}_r g_2)(x)\hat{\varphi}_y(x)m'(x) \tag{19}$$

$$\frac{d}{dx}\left[\frac{g_1'(x)}{S'(x)}\hat{\psi}_z(x) - \frac{\hat{\psi}_z'(x)}{S'(x)}g_1(x)\right] = (\mathcal{G}_r g_1)(x)\hat{\psi}_z(x)m'(x). \tag{20}$$

Finally, if a pair z^*, y^* satisfying the first order necessary conditions (15) and (16) exists then

$$\lim_{x\uparrow y^*} F'_{z^*,y^*}(x) = g_1(y^*)\frac{\hat{\psi}'_{z^*}(y^*)}{\hat{\psi}_{z^*}(y^*)} + g_2(z^*)\frac{\hat{\varphi}'_{y^*}(y^*)}{\hat{\varphi}_{y^*}(z^*)} = g_1(y^*)\frac{\hat{\psi}'_{z^*}(y^*)}{\hat{\psi}_{z^*}(y^*)} - \frac{S'(y^*)}{\hat{\psi}_{z^*}(y^*)}\frac{Bg_2(z^*)}{\varphi(z^*)} = g_1'(y^*)$$

and

$$\lim_{x\downarrow z^*} F'_{z^*,y^*}(x) = g_2(z^*)\frac{\hat{\varphi}'_{y^*}(z^*)}{\hat{\varphi}_{y^*}(z^*)} + g_1(y^*)\frac{\hat{\psi}'_{z^*}(z^*)}{\hat{\psi}_{z^*}(y^*)} = g_2(z^*)\frac{\hat{\varphi}'_{y^*}(z^*)}{\hat{\varphi}_{y^*}(z^*)} + \frac{S'(z^*)}{\hat{\varphi}_{y^*}(z^*)}\frac{Bg_1(y^*)}{\psi(y^*)} = g_2'(z^*)$$

completing the proof of our Lemma. □

Lemma 4.1 states the ordinary first order conditions which a pair of boundaries has to satisfy in order to constitute an extreme point. As usually in optimal stopping theory, we again find that since the exercise payoffs are differentiable at the optimal exercise thresholds the standard smooth-fit condition is satisfied in the present case as well (cf. Salminen (1985), Alvarez (2001)). Lemma 4.1 also presents an alternative formulation of the optimality conditions in terms of an integral representation based on the minimal r-harmonic mappings $\hat{\psi}_{z^*}(x)$ and $\hat{\varphi}_{y^*}(x)$ for the underlying diffusion killed at the boundaries z^* and y^* (i.e. the killed diffusion $\{X_t; t \in [0, \lambda_{z^*} \wedge \lambda_{y^*})\}$). An interesting implication of the observations of Lemma 4.1 expressing the value as well as the stopping boundaries directly in terms of the minimal r-harmonic mappings for the underlying diffusion $\{X_t; t \in [0, \zeta)\}$ is now summarized in the following corollary.

Corollary 4.2. *Assume that the exercise payoffs are differentiable at the optimal boundaries z^* and y^*. Then the optimality conditions* (15) *and* (16) *can be re-expressed as $(L_\varphi g_2)(z^*) = (L_\varphi g_1)(y^*)$ and $(L_\psi g_2)(z^*) = (L_\psi g_1)(y^*)$. Moreover, in that case $F_{z^*,y^*}(x) = c_1\psi(x) + c_2\varphi(x)$, where $c_1 = B^{-1}(L_\varphi g_2)(z^*) = B^{-1}(L_\varphi g_1)(y^*)$ and $c_2 = -B^{-1}(L_\psi g_2)(z^*) = -B^{-1}(L_\psi g_1)(y^*)$.*

Proof. The value $F_{z^*,y^*}(x)$ satisfies the ordinary differential equation $(\mathcal{G}_r F_{z^*,y^*})(x) = 0$ on the continuation region (z^*, y^*) where the underlying diffusion is not stopped. Thus, it can be expressed in terms of the fundamental solutions as $F_{z^*,y^*}(x) = c_1\psi(x) + c_2\varphi(x)$, where c_1, c_2 are unknown constants. Combining the value-matching with the smooth-fit condition at the lower boundary z^* yields the system of equations

$$F_{z^*,y^*}(z^*) = c_1\psi(z^*) + c_2\varphi(z^*) \quad = \quad g_2(z^*) \tag{21}$$

$$F'_{z^*,y^*}(z^*) = c_1\psi'(z^*) + c_2\varphi'(z^*) \quad = \quad g_2'(z^*) \tag{22}$$

Solving the unknown constants c_1 and c_2 from (21) and (22) yields

$$c_1 = B^{-1}\left[\frac{g_2'(z^*)}{S'(z^*)}\varphi(z^*) - \frac{\varphi'(z^*)}{S'(z^*)}g_2(z^*)\right] = B^{-1}(L_\varphi g_2)(z^*) \tag{23}$$

and

$$c_2 = B^{-1}\left[\frac{\psi'(z^*)}{S'(z^*)}g_2(z^*) - \frac{g_2'(z^*)}{S'(z^*)}\psi(z^*)\right] = -B^{-1}(L_\psi g_2)(z^*). \tag{24}$$

Analogously, combining the value-matching with the smooth-fit condition at the upper boundary and solving the resulting linear equation yields

$$c_1 = B^{-1}\left[\frac{g_1'(y^*)}{S'(y^*)}\varphi(y^*) - \frac{\varphi'(y^*)}{S'(y^*)}g_1(y^*)\right] = B^{-1}(L_\varphi g_1)(y^*) \tag{25}$$

and

$$c_2 = B^{-1}\left[\frac{\psi'(y^*)}{S'(y^*)}g_1(y^*) - \frac{g_1'(y^*)}{S'(y^*)}\psi(y^*)\right] = -B^{-1}(L_\psi g_1)(y^*). \tag{26}$$

Combining (23) with (25) and (24) with (26) then proves the alleged claim. □

Corollary 4.2 demonstrates how the first order optimality conditions and the resulting value function can be expressed in terms of the functionals $(L_\psi g_i)(x)$ and $(L_\varphi g_i)(x)$ associated to the integral representation of r-excessive functions for the diffusion $\{X_t; t \in [0,\zeta)\}$ (cf. Salminen (1985)).

Our analysis above does not characterize how strategic interaction affects the optimal boundaries in comparison with the non-strategic case considered in the previous section. A set of relatively general conditions under which strategic interaction unambiguously speeds up exercise in comparison with the nonstrategic setting is now summarized in the following.

Theorem 4.3. *Assume that there is a pair of thresholds $\tilde{x}_i$, $i = 1, 2$ such that $(\mathcal{G}_r g_i)(x) \gtreqless 0$ for $x \lesseqgtr \tilde{x}_i$, $i = 1, 2$, and that a pair $(z^*, y^*) \in \mathcal{I}^2$ satisfying the optimality conditions* (17) *and* (18) *exists. Then, the pair $(z^*, y^*) \in \mathcal{I}^2$ is unique and $\hat{x}_2 < z^* < \tilde{x}_2$ and $\tilde{x}_1 < y^* < x_1^*$. Moreover, the value of the game reads as*

$$V(x) = \begin{cases} g_1(x) & x \in [y^*, b) \\ F_{z^*,y^*}(x) & x \in (z^*, y^*) \\ g_2(x) & x \in (a, z^*] \end{cases} \tag{27}$$

Proof. Consider for any fixed $y \in \mathcal{I}$ the functional

$$L_1(z) = \frac{B}{\psi(y)}(g_2(y) - g_1(y)) - \int_z^y (\mathcal{G}_r g_2)(x)\hat{\varphi}_y(x)m'(x)dx.$$

It is clear that $L_1(y) > 0$ for any $y \in \mathcal{I}$ and that $L_1(z)$ is increasing on $(a, \tilde{x}_2)$ and decreasing on $(\tilde{x}_2, b)$. Thus, if a root $z_y^* \in (a, y)$ satisfying the condition $L_1(z_y^*) = 0$ exists, it has to be on the set $(a, \tilde{x}_2)$. Analogously, consider for any fixed $z \in \mathcal{I}$ the functional

$$L_2(y) = \frac{B}{\varphi(z)}(g_1(z) - g_2(z)) - \int_z^y (\mathcal{G}_r g_1)(x)\hat{\psi}_z(x)m'(x)dx.$$

We observe that $L_2(z) < 0$ for any $z \in \mathcal{I}$, that $L_2(y)$ is decreasing on $(a, \tilde{x}_1)$, and that $L_2(y)$ is increasing on $(\tilde{x}_1, b)$. Consequently, if a root $y_z^* \in (z, b)$ satisfying the condition $L_2(y_z^*) = 0$ exists, it has to be on the set $(\tilde{x}_1, b)$. On the other hand, as was established in Corollary 4.2, the optimality conditions can be expressed as $(L_\varphi g_2)(z^*) = (L_\varphi g_1)(y^*)$ and $(L_\psi g_2)(z^*) = (L_\psi g_1)(y^*)$. In light of our observations on the functional $(L_\varphi g_1)(x)$ we find that

$$(L_\varphi g_1)(y^*) = (L_\varphi g_1)(y) - \int_{y^*}^{y} (\mathcal{G}_r g_1)(t)\varphi(t)m'(t)dt > 0$$

for any $\tilde{x}_1 < y^* < y < b$ since $g_1(x)$ in non-decreasing and $(\mathcal{G}_r g_1)(x) \leq 0$ on $(\tilde{x}_1, b)$. Hence, we find that $(L_\varphi g_2)(z^*) = (L_\varphi g_1)(y^*) > 0$. This, in turn, implies that $z^* > \hat{x}_2$ since $(L_\varphi g_2)(\hat{x}_2) = 0$. Establishing that $y^* < x_1^*$ is completely analogous.

We now establish that if a pair satisfying the first order conditions exists then it is unique. By differentiating implicitly the functionals $E_\varphi(z, y) = (L_\varphi g_2)(z) - (L_\varphi g_1)(y)$ and $E_\psi(z, y) = (L_\psi g_2)(z) - (L_\psi g_1)(y)$ we find that

$$\begin{aligned}
\frac{dy}{dz}\Big|_{E_\varphi(z,y)=0} &= \frac{(\mathcal{G}_r g_2)(z)\varphi(z)m'(z)}{(\mathcal{G}_r g_1)(y)\varphi(y)m'(y)} < 0 \\
\frac{dy}{dz}\Big|_{E_\psi(z,y)=0} &= \frac{(\mathcal{G}_r g_2)(z)\psi(z)m'(z)}{(\mathcal{G}_r g_1)(y)\psi(y)m'(y)} < 0
\end{aligned}$$

for all $(z, y) \in (a, \tilde{x}_2) \times (\tilde{x}_1, b)$. Since

$$\frac{dy}{dz}\Big|_{E_\varphi(z,y)=0} = \left(\frac{\psi(y)\varphi(z)}{\psi(z)\varphi(y)}\right) \frac{dy}{dz}\Big|_{E_\psi(z,y)=0} < \frac{dy}{dz}\Big|_{E_\psi(z,y)=0}$$

we find that if an interception point (z^*, y^*) exists, it is unique.

It remains to prove that the value of the game reads as in (27). To accomplish this task, we first assume that $x \in (z^*, y^*)$ and define the functionals

$$\begin{aligned}
\Delta_1(x) &= g_2(z^*)\frac{\hat{\varphi}_{y^*}(x)}{\hat{\varphi}_{y^*}(z^*)} + g_1(y^*)\frac{\hat{\psi}_{z^*}(x)}{\hat{\psi}_{z^*}(y^*)} - g_1(x) \\
\Delta_2(x) &= g_2(z^*)\frac{\hat{\varphi}_{y^*}(x)}{\hat{\varphi}_{y^*}(z^*)} + g_1(y^*)\frac{\hat{\psi}_{z^*}(x)}{\hat{\psi}_{z^*}(y^*)} - g_2(x)
\end{aligned}$$

which in light of our findings satisfy the conditions $\Delta_1(y^*) = \Delta_1'(y^*) = 0$ and $\Delta_2(z^*) = \Delta_2'(z^*) = 0$. Standard differentiation yields

$$\begin{aligned}
\frac{d}{dx}\left[\frac{\Delta_1(x)}{\hat{\varphi}_{y^*}(x)}\right] &= \frac{S'(x)}{\hat{\varphi}_{y^*}^2(x)}\left[\frac{Bg_1(y^*)}{\psi(y^*)} - \frac{g_2'(x)}{S'(x)}\hat{\varphi}_{y^*}(x) + \frac{\hat{\varphi}_{y^*}'(x)}{S'(x)}g_2(x)\right] \\
\frac{d}{dx}\left[\frac{\Delta_2(x)}{\hat{\psi}_{z^*}(x)}\right] &= \frac{S'(x)}{\hat{\psi}_{z^*}^2(x)}\left[-\frac{Bg_2(z^*)}{\varphi(z^*)} - \frac{g_1'(x)}{S'(x)}\hat{\psi}_{z^*}(x) + \frac{\hat{\psi}_{z^*}'(x)}{S'(x)}g_1(x)\right].
\end{aligned}$$

Invoking the optimality conditions (15) and (16) and applying the observations (19) and

(20) then implies that

$$\begin{aligned}\frac{d}{dx}\left[\frac{\Delta_1(x)}{\hat{\varphi}_{y^*}(x)}\right] &= -\frac{S'(x)}{\hat{\varphi}^2_{y^*}(x)}\int_{z^*}^{x}\hat{\varphi}_{y^*}(t)(\mathcal{G}_r g_2)(t)m'(t)dt < 0\\ \frac{d}{dx}\left[\frac{\Delta_2(x)}{\hat{\psi}_{z^*}(x)}\right] &= \frac{S'(x)}{\hat{\psi}^2_{z^*}(x)}\int_{x}^{y^*}\hat{\psi}_{z^*}(t)(\mathcal{G}_r g_1)(t)m'(t)dt < 0\end{aligned}$$

since $x \in (z^*, y^*)$ and $a < z^* < \tilde{x}_2$ and $\tilde{x}_1 < y^* < b$. Thus, we observe that $\Delta_1(x) \geq \Delta_1(y^*) = 0$ and $\Delta_2(x) \leq \Delta_2(z^*) = 0$ for all $x \in (z^*, y^*)$. Consequently, $g_1(x) \leq V(x) \leq g_2(x)$ for all $x \in (z^*, y^*)$. Moreover, since $z^* < \tilde{x}_2$ and $y^* > \tilde{x}_1$ we observe that the proposed value function is r-superharmonic on (z^*, b) where the value is strictly smaller than $g_2(x)$ and r-subharmonic on the set (a, y^*) where the value is strictly larger than $g_1(x)$. Hence, z^*, y^* defines a saddle point strategy and $V(x)$ constitutes the value of the game. □

Theorem 4.3 demonstrates that if an extremal pair characterizing the saddle point solution exists, then this pair is unique and strategic interaction accelerates exercise in comparison with non-strategic setting as is characterized by the inclusion $(z^*, y^*) \subset (\hat{x}_2, x_1^*)$. This observation is of interest since it does not depend on the particular boundary behavior of the underlying diffusion and is valid whenever a pair satisfying the ordinary first order conditions exists. Unfortunately, the conditions of Theorem 4.3 are not sufficient for the existence of a pair z^*, y^* defining a saddle point strategy and, therefore, more analysis is needed. A set of sufficient conditions under which the conclusions of our previous theorem holds are now summarized in the following.

Theorem 4.4. *Assume that the boundaries a and b are natural for the diffusion $\{X_t; t \geq 0\}$, and that*

(i) $\mathcal{G}_r g_i \in \mathcal{L}^1$ and $\lim_{x\downarrow a}(L_\psi g_i)(x) = \lim_{x\uparrow b}(L_\varphi g_i)(x) = 0$ for $i = 1, 2$,

(ii) there is a pair of thresholds $\tilde{x}_i$, $i = 1, 2$ such that $(\mathcal{G}_r g_i)(x) \gtreqless 0$ for $x \lesseqgtr \tilde{x}_i$, and

(iii) $(\mathcal{G}_r g_1)(x) > (\mathcal{G}_r g_2)(x)$ for all $x \in \mathcal{I}\backslash\{y_j\}_{j\in I}$.

Then a unique pair (z^*, y^*) satisfying the first order conditions exists and the value of the game reads as in (27).

Proof. Assumption (i) implies that

$$(L_\psi g_i)(x) = \int_a^x \psi(t)(\mathcal{G}_r g_i)(t)m'(t)dt$$

and

$$(L_\varphi g_i)(x) = -\int_x^b \varphi(t)(\mathcal{G}_r g_i)(t)m'(t)dt.$$

Consequently, we observe that the first order conditions $(L_\varphi g_2)(z^*) = (L_\varphi g_1)(y^*)$ and $(L_\psi g_2)(z^*) = (L_\psi g_1)(y^*)$ can be re-expressed as

$$\begin{aligned}\int_z^y \psi(t)(\mathcal{G}_r g_1)(t)m'(t)dt &= \int_a^z \psi(t)[(\mathcal{G}_r g_2)(t) - (\mathcal{G}_r g_1)(t)]m'(t)dt < 0 &\quad (28)\\ \int_z^y \varphi(t)(\mathcal{G}_r g_2)(t)m'(t)dt &= \int_y^b \varphi(t)[(\mathcal{G}_r g_1)(t) - (\mathcal{G}_r g_2)(t)]m'(t)dt > 0. &\quad (29)\end{aligned}$$

Consider now the behavior of the functionals

$$\begin{aligned}K_1(y) &= \int_z^y \psi(t)(\mathcal{G}_r g_1)(t)m'(t)dt\\ K_2(z) &= \int_z^y \varphi(t)(\mathcal{G}_r g_2)(t)m'(t)dt.\end{aligned}$$

It is clear that our assumptions imply that if $z \in \mathcal{I}$ and $y \in (\tilde{x}_1 \vee z, b)$ is arbitrary then the mean value theorem for integrals implies that

$$K_1(y) = K_1(\tilde{x}_1 \vee z) + \int_{\tilde{x}_1 \vee z}^y \psi(t)(\mathcal{G}_r g_1)(t)m'(t)dt = K_1(\tilde{x}_1 \vee z) + \frac{(\mathcal{G}_r g_1)(\xi_1)}{r}\left(\frac{\psi'(y)}{S'(y)} - \frac{\psi'(\tilde{x}_1 \vee z)}{S'(\tilde{x}_1 \vee z)}\right)$$

where $\xi_1 \in (\tilde{x}_1 \vee z, b)$. Since $\psi'(y)/S'(y) \uparrow \infty$ as $y \uparrow b$ and $(\mathcal{G}_r g_1)(\xi_1) < 0$ we find that $K_1(y) \downarrow -\infty$ as $y \uparrow b$. Hence, this observation demonstrates that equation (28) has for any $z \in \mathcal{I}$ a unique root $y_z^* \in (z \vee \tilde{x}_1, b)$. Analogously, our assumptions imply that if $y \in \mathcal{I}$ is given and $z \in (a, \tilde{x}_2 \wedge y)$ is arbitrary then the mean value theorem for integrals implies that

$$K_2(z) = K_2(\tilde{x}_2 \wedge y) + \int_z^{\tilde{x}_2 \wedge y} \varphi(t)(\mathcal{G}_r g_2)(t)m'(t)dt = K_2(\tilde{x}_2 \wedge y) + \frac{(\mathcal{G}_r g_2)(\xi_2)}{r}\left(\frac{\varphi'(\tilde{x}_2 \wedge y)}{S'(\tilde{x}_2 \wedge y)} - \frac{\varphi'(z)}{S'(z)}\right)$$

where $\xi_2 \in (z, \tilde{x}_2 \wedge y)$. Since $\varphi'(z)/S'(z) \downarrow -\infty$ as $z \downarrow a$ and $(\mathcal{G}_r g_2)(\xi_2) > 0$ we find that $K_2(z) \uparrow \infty$ as $z \downarrow a$ and, therefore, that equation (29) has for any $y \in \mathcal{I}$ a unique root $z_y^* \in (z, \tilde{x}_2 \wedge y)$. On the other hand, conditions (ii) and (iii) imply that $\tilde{x}_1 > \tilde{x}_2$. Consequently, we find that if a pair (z^*, y^*) satisfying the first order conditions (28) and (29) exists, it is necessarily on the set $(a, \tilde{x}_2) \times (\tilde{x}_1, b)$. Thus, in light of our previous uniqueness proof it is sufficient to establish that the curves z^*_y and y^*_z intercept on $(a,\tilde{x}_2)\times(\tilde{x}_1,b)$. To accomplish this task, we first observe from Lemma 3.1 that $z^*_b=\hat{x}_2$ and $y^*_a=x^*_1$. Moreover, the boundary properties of the mappings $K_1(y)$ and $K_2(z)$ guarantee that $z^*_{\tilde{x}_1}<\tilde{x}_2$ and $y^*_{\tilde{x}_2}>\tilde{x}_1$. Hence, monotonicity implies that $y^*_z:(a,\tilde{x}_2)\mapsto(y^*_{\tilde{x}_2},x^*_1)$, $z^*_y:(\tilde{x}_1,b)\mapsto(\hat{x}_2,z^*_{\tilde{x}_1})$, $y^*_{z^*_b}=y^*_{\hat{x}_2}<x^*_1=y^*_a<b$, and $y^*_{z^*_{\tilde{x}_1}}>y^*_{\tilde{x}_2}>\tilde{x}_1>\tilde{x}_2>z^*_{\tilde{x}_1}$. Consequently, we observe that the mappings z^*_y and y^*_z have an interception point on $(a,\tilde{x}_2)\times(\tilde{x}_1,b)$ from which the alleged existence and uniqueness of a solution follows. The rest of our conclusions now follow directly from Theorem 4.3. □

Theorem 4.4 states a set of conditions under which a unique pair (z^*,y^*) satisfying the first order conditions exists and under which the value of the game can be expressed as in (27). As is clear from the proof of our theorem, the boundary behavior of the underlying process plays a key role in the proof of the existence of a saddle point equilibrium by determining the behavior of the associated functionals at the boundaries of the state space. Naturally, establishing the existence of a saddle point equilibrium in a more general setting requires a stronger set of sufficient conditions.

Even though the findings of Theorem 4.4 and Theorem 4.3 are applicable within a relatively large class of problems our assumptions on the roots (and, therefore, the sign) of the functions $(\mathcal{G}_r g_i)(x)$ are not always satisfied in cases where the exercise payoffs have option characteristics (for example, in the call option case where $g_i(x)=(x-c_i)^+$, $c_1>c_2>0$). A set of results characterizing the saddle point solution for a class of problems of this type are now summarized in the following.

Corollary 4.5. *Assume that there exists $a<\bar{x}_i<\tilde{x}_i<b$ so that $g_i(x)=0$ on $(0,\bar{x}_i)$, $(\mathcal{G}_r g_i)(x)>0$ on $(\bar{x}_i,\tilde{x}_i)$, and $(\mathcal{G}_r g_i)(x)<0$ on $(\tilde{x}_i,b)$, $i=1,2$, and that $\bar{x}_1\geq\bar{x}_2$. Assume also that the threshold*

$$y^*_{\bar{x}_2}=\underset{y}{\operatorname{argmax}}\left\{\frac{g_1(y)}{\hat{\psi}_{\bar{x}_2}(y)}\right\}$$

exists. If a pair $(z^*,y^*)\in(\bar{x}_2,\tilde{x}_2)\times(\tilde{x}_1,y^*_{\bar{x}_2})$ satisfying the first order conditions (15) *and* (16) *exists, then the conclusions of Theorem 4.3 are satisfied and the value of the game reads as in* (27).

Proof. The alleged result is a direct implication of Theorem 4.3 after noticing that $y^*_{\bar{x}_2}$ constitutes the corner solution of (16). □

Corollary 4.5 states a set of condition under which the conclusions of our Theorem 4.3 are satisfied within a more general setting as well. It is clear from Corollary 4.5 that $y^*_{\bar{x}_2}$ constitutes the maximal solution for the upper boundary and, therefore, that there are circumstances under which the first order conditions (15) and (16) do not necessarily admit an interior solution. In that case, it is naturally of interest to investigate under which conditions (on the parameters of the problem) the pair of equations

$$\frac{g_1'(y^*_{\bar{x}_2})}{S'(y^*_{\bar{x}_2})}\psi(y^*_{\bar{x}_2}) - \frac{\psi'(y^*_{\bar{x}_2})}{S'(y^*_{\bar{x}_2})}g_1(y^*_{\bar{x}_2}) = \lim_{z\downarrow\bar{x}_2}\frac{g_2'(z)}{S'(z)}\psi(z) \tag{30}$$

$$\frac{g_1'(y^*_{\bar{x}_2})}{S'(y^*_{\bar{x}_2})}\varphi(y^*_{\bar{x}_2}) - \frac{\varphi'(y^*_{\bar{x}_2})}{S'(y^*_{\bar{x}_2})}g_1(y^*_{\bar{x}_2}) = \lim_{z\downarrow\bar{x}_2}\frac{g_2'(z)}{S'(z)}\varphi(z) \tag{31}$$

has a solution. It is clear that if this solution exists, then it characterizes the extreme circumstances under which the considered game has a saddle point equilibrium and a value

$$V(x) = \begin{cases} g_1(x) & x \geq y^*_{\bar{x}_2} \\ g_1(y^*_{\bar{x}_2})\frac{\hat{\psi}_{\bar{x}_2}(x)}{\hat{\psi}_{\bar{x}_2}(y^*_{\bar{x}_2})} & \bar{x}_2 < x < y^*_{\bar{x}_2} \\ 0 & x \leq \bar{x}_2 \end{cases}$$

which satisfies both the value matching $V(y^*_{\bar{x}_2}) = g_1(y^*_{\bar{x}_2})$ and $V(\bar{x}_2+) = 0$ as well as the smooth fit conditions $V'(y^*_{\bar{x}_2}) = g_1'(y^*_{\bar{x}_2})$ and $V'(\bar{x}_2+) = g_2'(\bar{x}_2+)$ at the exercise boundaries. An economically interesting implication of this observation is that if increased volatility expands the continuation region by lowering the threshold z^* and increasing the threshold y^* then the first order conditions (30) and (31) characterize the extremal combination of boundaries for a given prevailing volatility level. Consequently, if volatility increases in that case then the saddle point equilibrium is lost. Under such circumstances the game has a value and a well-defined saddle point equilibrium only for sufficiently volatilities below this critical level. We will illustrate this observation later in an explicitly parametrized example based on geometric Brownian motion. Naturally, other parametric changes resulting into a similar reaction lead to a similar situation where the saddle point equilibrium strategy is abolished.

5 Explicit Illustrations

5.1 Smooth Payoff Case

In order to illustrate our general findings in an explicitly solvable framework, we now consider the stopping game characterized by the exercise payoff

$$\Pi_x(\tau,\gamma) = \mathbf{E}_x\left[e^{-r(\tau\wedge\gamma)}\left(((R_rf)(X_\tau)-c_1)\,1_{\{\tau\leq\gamma\}}+((R_rf)(X_\gamma)-c_2)\,1_{\{\tau>\gamma\}}\right)\right] \tag{32}$$

where $c_1 > c_2 > 0$ are exogenously determined constants measuring the sunk costs associated to the timing decision, $f \in \mathcal{L}^1$ is a continuous and non-decreasing profit flow satisfying the conditions $\lim_{x\downarrow a} f(x) < rc_2 < rc_1 < \lim_{x\to b} f(x)$, and X_t evolves according to the stochastic dynamics characterized by the stochastic differential equation (1). For simplicity, we also assume that the boundaries of the state-space of the diffusion are natural and, therefore, that the valuation has an infinite time horizon.

It is worth observing that since the buyer gets in the present example always the expected cumulative present value $(R_rf)(x)$, the only variable factor which depends on the precise timing of the decision is the cost which the buyer incurs (and the seller receives) at exercise. Thus, the considered game can be interpreted as the valuation of an investment opportunity which guarantees the buyer a permanent flow of revenues from the exercise date up to an arbitrarily distant future at a cost which is endogenously determined from the game.

It is clear that the exercise payoffs can be re-expressed as $g_i(x) = (R_rf)(x) - c_i = (R_r\pi_i)(x)$, where $\pi_i(x) = f(x) - rc_i$. Thus, applying (2) implies that in the present example we have

$$\begin{aligned}(L_\varphi g_i)(x) &= \int_x^b \varphi(v)(f(v)-rc_i)m'(v)dv\\ (L_\psi g_i)(x) &= -\int_a^x \psi(v)(f(v)-rc_i)m'(v)dv.\end{aligned}$$

Since $(\mathcal{G}_r g_i)(x) = rc_i - f(x)$, we observe that the conditions of our Lemma 3.1 are satisfied and, therefore, that in the absence of strategic interaction there are two stopping boundaries $\hat{x}_2$ and x_1^* satisfying the optimality conditions $(L_\varphi g_2)(\hat{x}_2) = 0$ and $(L_\psi g_1)(x_1^*) = 0$. In that case $\hat{x}_2$ constitutes the optimal boundary at which the expected present value of the payoff

$g_2(x)$ is minimized and x_1^* constitutes the optimal boundary at which the expected present value of the payoff $g_1(x)$ is maximized. Moreover, we also observe that if a pair z^*, y^* of optimal boundaries exist, it has to satisfy the optimality conditions $(L_\varphi g_2)(z^*) = (L_\varphi g_1)(y^*)$ and $(L_\psi g_2)(z^*) = (L_\psi g_1)(y^*)$ which can be re-expressed as

$$\int_{z^*}^{y^*} \varphi(t)(f(v) - rc_2)m'(v)dv - (c_1 - c_2)\frac{\varphi'(y^*)}{S'(y^*)} = 0 \tag{33}$$

$$\int_{z^*}^{y^*} \psi(v)(f(v) - rc_1)m'(v)dv + (c_2 - c_1)\frac{\psi'(z^*)}{S'(z^*)} = 0. \tag{34}$$

Our main conclusions on this problem are now summarized in the following theorem.

Theorem 5.1. *There is a unique pair z^* and y^* satisfying the optimality conditions* (33) *and* (34). *The value of the game is continuously differentiable and reads as*

$$V(x) = \begin{cases} (R_r\pi_1)(x) & x \in [y^*, b) \\ (R_r\pi_2)(z^*)\frac{\hat{\varphi}_{y^*}(x)}{\hat{\varphi}_{y^*}(z^*)} + (R_r\pi_1)(y^*)\frac{\hat{\psi}_{z^*}(x)}{\hat{\psi}_{z^*}(y^*)} & x \in (z^*, y^*) \\ (R_r\pi_2)(x) & x \in (a, z^*]. \end{cases} \tag{35}$$

Moreover, $f(\hat{x}_2) < f(z^*) < rc_2$ and $rc_1 < f(y^*) < f(x_1^*)$.

Proof. We first establish the existence and uniqueness of the pair z^* and y^* satisfying the optimality conditions (33) and (34). To this end, consider the mappings

$$P_1(z, y) = \int_z^y \varphi(v)(f(v) - rc_2)m'(v)dv - (c_1 - c_2)\frac{\varphi'(y)}{S'(y)} \tag{36}$$

$$P_2(z, y) = \int_z^y \psi(v)(f(v) - rc_1)m'(v)dv + (c_2 - c_1)\frac{\psi'(z)}{S'(z)}. \tag{37}$$

It is clear that $P_1(y, y) = -(c_1 - c_2)\varphi'(y)/S'(y) > 0$ for all $y \in \mathcal{I}$ and $P_1(z, y) > 0$ whenever $(z, y) \subset \{t \in \mathcal{I} : f(t) \geq rc_2\}$. Thus, equation $P_1(z, y) = 0$ can have a root only if $z < \inf\{t : f(t) \geq rc_2\}$. Let $z < k < \inf\{t : f(t) = rc_2\}$. Then the monotonicity of the mapping $f(t) - rc_2$ implies that

$$P_1(z, y) = P_1(k, y) + \int_z^k \varphi(v)(f(v) - rc_2)m'(v)dv \leq P_1(k, y) + \frac{f(k) - rc_2}{r}\left(\frac{\varphi'(k)}{S'(k)} - \frac{\varphi'(z)}{S'(z)}\right).$$

Since $\varphi'(z)/S'(z) \downarrow -\infty$ as $z \downarrow a$ we find that $\lim_{z\downarrow a} P_1(z, y) = -\infty$ for all $y \in \mathcal{I}$ and, therefore, that equation $P_1(z, y) = 0$ has for any $y \in \mathcal{I}$ a root $z_y^* \in \{t \in \mathcal{I} : f(t) < rc_2\}$.

Moreover, standard implicit differentiation yields

$$z_y^{*\prime} = \frac{\varphi(y)(f(y) - rc_1)m'(y)}{\varphi(z_y^*)(f(z_y^*) - rc_2)m'(z_y^*)} \gtreqless 0, \quad f(y) \lesseqgtr rc_1 \tag{38}$$

implying that z_y^* attains its maximum value on the set where $f(y) = rc_1$.

Consider now the mapping $P_2(z, y)$. It is clear that $P_2(z, z) = (c_2 - c_1)\psi'(z)/S'(z) < 0$ for all $z \in \mathcal{I}$ and that $P_2(z, y) < 0$ as long as $(z, y) \subset \{t \in \mathcal{I} : f(t) \leq rc_1\}$. Thus, we observe that equation $P_2(z, y) = 0$ can have a root only if $y > \sup\{t \in \mathcal{I} : f(t) \leq rc_1\}$. Let $y > k > \sup\{t \in \mathcal{I} : f(t) = rc_1\}$. Again, the monotonicity of the mapping $f(t) - rc_1$ implies that

$$P_2(z, y) = P_2(z, k) + \int_k^y \psi(v)(f(v) - rc_1)m'(v)dv \geq P_2(z, k) + \frac{(f(k) - rc_1)}{r}\left(\frac{\psi'(y)}{S'(y)} - \frac{\psi'(k)}{S'(k)}\right).$$

Since $\psi'(y)/S'(y) \uparrow \infty$ as $y \uparrow b$ we find that $\lim_{y \uparrow b} P_2(z, y) = \infty$ for all $z \in \mathcal{I}$ and, therefore, that equation $P_2(z, y) = 0$ has for any $z \in \mathcal{I}$ a root $y_z^* \in \{t \in \mathcal{I} : f(t) > rc_1\}$. Standard implicit differentiation now yields

$$y_z^{*\prime} = \frac{\psi(z)(f(z) - rc_2)m'(z)}{\psi(y_z^*)(f(y_z^*) - rc_1)m'(y_z^*)} \gtreqless 0, \quad f(z) \gtreqless rc_2 \tag{39}$$

showing that y_z^* attains its minimum on the set where $f(z) = rc_2$.

Combining these observations clearly indicate that if an optimal pair z^*, y^* satisfying the optimality conditions (33) and (34) exist, we necessarily have that $f(z^*) < rc_2$ and $f(y^*) > rc_1$. Moreover, the inequalities (38) and (39) imply that for $(z, y) \in (a, f^{-1}(rc_2)) \times (f^{-1}(rc_1), b)$ we have $f^{-1}(rc_2) > z_{f^{-1}(rc_1)}^* > z_y^* > z_b^* > a$ and $b > y_a^* > y_z^* > y_{f^{-1}(rc_2)}^* > f^{-1}(rc_1)$. Combining these observations with the monotonicity and continuity of the solutions z_y^* and y_z^* (by inequalities (38) and (39)) then imply that z_y^* and y_z^* have at least one interception point on $\{(z, y) \in \mathcal{I}^2 : z \in (a, f^{-1}(rc_2)), y \in (f^{-1}(rc_1), b)\}$. Since

$$\frac{d}{dz}\left[\frac{\psi(z)}{\varphi(z)}\right] = \frac{BS'(z)}{\varphi^2(z)} > 0$$

uniqueness follows from the observation

$$\left.\frac{dy}{dz}\right|_{P_2(z,y)=0} = \left[\frac{\psi(z)\varphi(y)}{\psi(y)\varphi(z)}\right] \left.\frac{dy}{dz}\right|_{P_1(z,y)=0} > \left.\frac{dy}{dz}\right|_{P_1(z,y)=0}$$

for all $(z, y) \in (a, f^{-1}(rc_2)) \times (f^{-1}(rc_1), b)$.

Having established the existence and uniqueness of an optimal pair, we now prove that the proposed value function is indeed the value of the game. We first observe that the proposed value function is continuously differentiable on $\mathcal{I}$, twice continuously differentiable on $\mathcal{I}\backslash(\{z^*\}\cup\{y^*\})$, $|V''(z^*\pm)|<\infty$, and $|V''(y^*\pm)|<\infty$. Moreover, $(\mathcal{G}_rV)(x)=0$ on (z^*,y^*), $(\mathcal{G}_rV)(x)=rc_2-f(x)>0$ on (a,z^*) and $(\mathcal{G}_rV)(x)=rc_1-f(x)<0$ on (y^*,b). Thus, the proposed value function is r-subharmonic on the set (a,y^*) where it strictly majorizes $(R_r\pi_1)(x)$ and r-superharmonic on the set (z^*,b) where it strictly minorizes $(R_r\pi_2)(x)$. This proves that the proposed value is indeed the value of the game.

The inequalities $f(\hat{x}_2)<f(z^*)<rc_2$ and $rc_1<f(y^*)<f(x_1^*)$ are direct implications of our analysis above, the monotonicity of $f(t)$, and the identities $(L_\varphi g_2)(\hat{x}_2)=0$ and $(L_\psi g_1)(x_1^*)=0$. □

In order to illustrate the findings of our Theorem 5.1 in an explicitly parametrized case assume now that $\mu(x)=\mu x$ and $\sigma(x)=\sigma x$, where $\mu,\sigma\in\mathbb{R}_+$ are known constants and that $f(x)=x^\theta$, where $\theta>0$ is an exogenously given constant. It is well-known that in this case the fundamental solutions of the ordinary second order differential equation $(\mathcal{G}_ru)(x)=0$ read as $\psi(x)=x^\eta$ and $\varphi(x)=x^\nu$, where

$$\eta=\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}>0$$

and

$$\nu=\frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}<0$$

denote the the roots of the characteristic equation $\sigma^2a(a-1)+2\mu=2r$. Moreover, if the absence of speculative bubbles condition $r>\theta\mu+\frac{1}{2}\sigma^2\theta(\theta-1)$ is satisfied then the expected cumulative present value of the cash flow x^θ exists and reads as

$$(R_rf)(x)=\frac{x^\theta}{r-\delta(\theta)},$$

where $\delta(\theta)=\theta\mu+\frac{1}{2}\sigma^2\theta(\theta-1)$ denotes the expected growth rate of the flow x^θ.

Given the observations above, let us now investigate the optimal stopping game and its value. Our assumption $r>\delta(\theta)$ implies that $\eta>\theta$ since $(\eta-\theta)(\theta-\nu)=2(r-\theta\mu-$

$\frac{1}{2}\sigma^2\theta(\theta-1))/\sigma^2$. Thus, the non-strategic exercise boundaries now read as

$$\hat{x}_2 = \left(\frac{\nu(r-\delta(\theta))c_2}{\nu-\theta}\right)^{1/\theta} = \left(\left(1-\frac{\theta}{\eta}\right)rc_2\right)^{1/\theta}$$

and

$$x_1^* = \left(\frac{\eta(r-\delta(\theta))c_1}{\eta-\theta}\right)^{1/\theta} = \left(\left(1-\frac{\theta}{\nu}\right)rc_1\right)^{1/\theta}.$$

Moreover, the optimality conditions defining the optimal boundaries z^* and y^* can be expressed as

$$\frac{\theta-\nu}{r-\delta(\theta)}z^{*\theta-\eta} + \nu c_2 z^{*-\eta} = \frac{\theta-\nu}{r-\delta(\theta)}y^{*\theta-\eta} + \nu c_1 y^{*-\eta} \tag{40}$$

$$\frac{\eta-\theta}{r-\delta(\theta)}z^{*\theta-\nu} - \eta c_2 z^{*-\nu} = \frac{\eta-\theta}{r-\delta(\theta)}y^{*\theta-\nu} - \eta c_1 y^{*-\nu}. \tag{41}$$

Unfortunately, solving the optimal boundaries from these equations explicitly is extremely difficult, if possible at all. Thus, we illustrate the optimal boundaries numerically in Table 1 (under the parameter specifications that $c_1 = 20, c_2 = 10, r = 0.04, \mu = 0.02$, and $\theta = 1$). It is worth noticing that our numerical results show that increased volatility expands the

Table 1: The Optimal Exercise Boundaries

σ	0.1	0.2	0.3	0.4	0.5
z^*	0.2137	0.1838	0.1603	0.1427	0.1294
y^*	0.9699	1.3355	1.7872	2.3145	2.9208
$\hat{x}_2$	0.1649	0.1172	0.0827	0.0597	0.0443
x_1^*	0.9702	1.3657	1.9346	2.6806	3.6114

continuation region by decreasing z^* and increasing y^*. This result is of interest since it indicates that the comparative static properties obtained in studies considering the valuation of investment opportunities in the non-strategic case can be extended to the strategic case as well at least in the present example. The value of the game is illustrated in Figure 1 (under the parameter specifications that $c_1 = 20, c_2 = 10, r = 0.04, \mu = 0.02, \sigma = 0.1$, and $\theta = 1$).

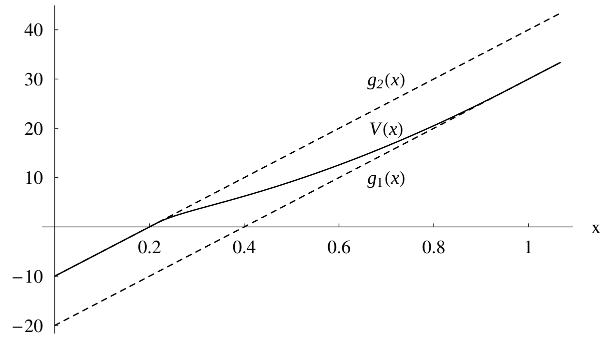

Figure 1: The Value Function and the Exercise Payoffs

For the sake of comparison, the value of the game in the case where the cash flow is concave is illustrated in Figure 2 (under the parameter specifications that $c_1 = 20, c_2 = 10, r = 0.04, \mu = 0.02, \sigma = 0.1$, and $\theta = 0.8$).

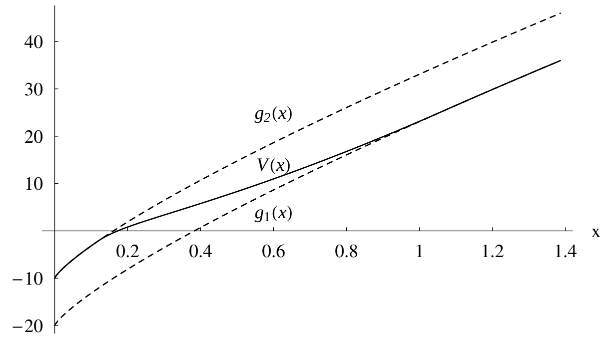

Figure 2: The Value Function and the Exercise Payoffs

5.2 Non-smooth Case

In order to illustrate a non-smooth case where a corner solution arises, assume now for simplicity that $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$, where $\mu, \sigma \in \mathbb{R}_+$ are known constants, that $r > \mu$, and that

$$g_i(x) = \left(\frac{x}{r-\mu} - c_i\right)^+,$$

where $c_1 > c_2 > 0$. Given this characterization, define the critical pair (σ^*, Y^*) as the solution of the equations

$$(1-\nu)Y^{*1-\eta} + \nu(r-\mu)c_1 Y^{*-\eta} = ((r-\mu)c_2)^{1-\eta} \tag{42}$$

$$(\eta-1)Y^{*1-\nu} - \eta(r-\mu)c_1 Y^{*-\nu} = ((r-\mu)c_2)^{1-\nu}, \tag{43}$$

where $\eta > 1$ and $\nu < 0$ are defined as in the previous section. In light of the first order optimality conditions (40) and (41) and the monotonicity of the root y_z^* characterizing the upper exercise threshold as a function of the lower boundary, it is clear that the equations (42) and (43) characterize the critical pair for which the value function satisfies the smooth fit conditions at the boundaries $z^* = (r-\mu)c_2 = g_2^{-1}(0)$ and $Y^* = y^*_{(r-\mu)c_2}$.

Given the above definition, we immediately observe that if $\sigma \leq \sigma^*$, then the first order optimality conditions (40) and (41) have an interior root (z^*, y^*) so that the resulting value function

$$V(x) = \begin{cases} g_1(x) & x \geq y^* \\ g_2(z^*)\frac{x^\nu - y^{*\nu-\eta}x^\eta}{z^{*\nu} - y^{*\nu-\eta}z^{*\eta}} + g_1(y^*)\frac{x^\eta - z^{*\eta-\nu}x^\nu}{y^{*\eta} - z^{*\eta-\nu}y^{*\nu}} & z^* < x < y^* \\ g_2(x) & x \leq z^* \end{cases}$$

satisfies the smooth fit conditions at both z^* and y^*. In that case the hitting times $\tau^* = \inf\{t \geq 0 : X_t \geq y^*\}$ and $\gamma^* = \inf\{t \geq 0 : X_t \leq z^*\}$ constitute a saddle point for the game. If, however, $\sigma > \sigma^*$, then the smooth fit condition cannot be satisfied at $(r-\mu)c_2$ and, thus, the game has no value. Naturally, in this case

$$\underline{V}(x) = \begin{cases} g_1(x) & x \geq y^*_{(r-\mu)c_2} \\ g_1(y^*_{(r-\mu)c_2})\frac{x^\eta - ((r-\mu)c_2)^{\eta-\nu}x^\nu}{{y^*_{(r-\mu)c_2}}^\eta - ((r-\mu)c_2)^{\eta-\nu}{y^*_{(r-\mu)c_2}}^\nu} & (r-\mu)c_2 < x < y^* \\ 0 & x \leq (r-\mu)c_2, \end{cases}$$

where

$$y^*_{(r-\mu)c_2} = \operatorname{argmax}\left\{\frac{g_1(x)}{x^\eta - ((r-\mu)c_2)^{\eta-\nu}x^\nu}\right\}.$$

We illustrate the optimal boundaries for various volatilities in Table 2 under the assumptions that $c_1 = 20, c_2 = 10, r = 0.035$, and $\mu = 0.02$ (which imply that $(\sigma^*, Y^*) \approx (0.212984, 1.25077)$). This finding is of interest since it shows that *increased volatility can*

Table 2: The Optimal Exercise Boundaries

σ	0.05	0.1	0.15	0.2
z^*	0.187	0.176	0.164	0.153
y^*	0.742	0.854	1.01	1.2
$\hat{x}_2$	0.142	0.123	0.103	0.0854
x_1^*	0.742	0.854	1.02	1.23

break up the existence of a saddle point equilibrium. Thus, within this modelling framework higher volatility does not only alter the equilibrium, it may very well abolish it.

6 Conclusions

In this paper we investigated a class of solvable Dynkin games by relying on an approach based on the classical theory of diffusions, stochastic calculus, and ordinary nonlinear programming techniques. Instead of analyzing all admissible stopping strategies at once, we restricted our interest to the strategies which can be characterized as the first exit time from an open set with compact closure on the state-space of the underlying diffusion and studied the value of these policies as functions of the arbitrary boundaries. We presented a set of conditions under which the value of the game can be explicitly derived in terms of the minimal r-harmonic mappings for the underlying diffusion and characterized the resulting saddle point equilibrium by applying the ordinary first order necessary conditions for the optimality of the arbitrary boundaries. Interestingly, our results indicated that there are circumstances under which parametric changes like increased volatility may break up the existence of a saddle point equilibrium.

There are several directions towards which our analysis could be naturally extended. A first natural extension would be to relax the zero-sum game assumption and analyze the Nash equilibrium for a class of time homogeneous nonzero-sum games. Given that in that case the value of the game can be analyzed by relying on quasi-variational inequalities (cf. Bensoussan and Friedman (1974, 1977)), the approach introduced in Alvarez (2004a,b) for solving functional recursions associated with quasi-variational inequalities could offer a promising technique for studying these games within a one-dimensional setting based on ordinary diffusions. A second interesting extension would be to analyze circumstances under which a broader class of strategies arise (along the lines of the study Touzi and Vieille (2002) where mixed strategies are analyzed). A third natural extension would be to increase the dimensionality of the stopping problem and analyze how this affects the equilibrium strategy. All these extensions are outside the scope of the present study and, therefore, left for future research.

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